High-Order Schemes for Backward Stochastic Differential Equations and Applications in Solving Partial-Integral Differential Equations

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Outline

Backward stochastic differential equations
driven by a Lèvy process

- Backward stochastic differential equation:

$$\begin{cases} X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \\ \qquad\qquad + \int_0^t \int_E c(s, X_{s-}, e)\tilde{\mu}(de, ds) \\ Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s, \Gamma_s)ds - \int_t^T Z_s dW_s \\ \qquad\qquad - \int_t^T \int_E U_s(e)\tilde{\mu}(de, ds), \end{cases}$$

where $E = \mathbb{R}^d \backslash \{0\}$, W_s is a d-dimensional Brownian motion and $\tilde{\mu}(de, ds)$ $= \mu(de, ds) - \lambda(de)ds$ is a compensated Poisson random measure.

- The solution is (X_t, Y_t, Z_t, U_t) and $\Gamma_t = \int_E U_t(e)\eta(e)\lambda(de)$.
- The well-posedness of the BSDE has been proved in [Pardoux-Peng,1990] for BSDEs driven by Brownian motion and in [Barles-Buckdahn-Pardoux, 1997] for BSDEs with jumps.

BSDEs and partial-integral differential equations (PIDEs)

stochastic representation of Kolmogorov backward equations (KBE)

We consider the viscosity solution $u(t,x) \in \mathcal{C}([0,T] \times \mathbb{R}^d)$ of the following PIDE, i.e.,

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \widetilde{\mathcal{L}}[u](t,x) + f(t,x,u,\sigma\nabla u, \mathcal{B}[u]) = 0, \text{ for } (t,x) \in [0,T) \times \mathbb{R}^d, \\ u(T,x) = \varphi(x), \text{ for } x \in \mathbb{R}^d, \end{cases}$$

where $\varphi(x)$ is the terminal condition at the time $t = T$ and the second-order integral-differential operator $\widetilde{\mathcal{L}}$ is of the form

$$\begin{aligned} \widetilde{\mathcal{L}}[u](t,x) = & \sum_{i=1}^{d} b_i(t,x) \frac{\partial u}{\partial x^i}(t,x) + \frac{1}{2} \sum_{i,j=1}^{d} (\sigma\sigma^\top)_{i,j}(t,x) \frac{\partial^2 u}{\partial x^i \partial x^j}(t,x) \\ & + \int_D \left(u(t, x + c(t,x,e)) - u(t,x) - \sum_{i=1}^{d} \frac{\partial u}{\partial x^i}(t,x) c(t,x,e) \right) \lambda(de), \end{aligned}$$

and $\mathcal{B}$ is an integral operator defined as

$$\mathcal{B}[u](t,x) = \int_D \big[u(t, x + c(t,x,e)) - u(t,x) \big] \eta(e) \lambda(de).$$

BSDEs and partial-integral differential equations (PIDEs)

stochastic representation of Kolmogorov backward equations

Under the condition that $X_t = x$ with $0 \le t \le T$, the conditional solution $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}, U_s^{t,x})$ for $t \le s \le T$ satisfies

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) dr + \int_t^s \sigma(r, X_r^{t,x}) dW_r + \int_t^s \int_E c(r, X_{r-}, e) \tilde{\mu}(de, dr), \\ Y_s^{t,x} = \varphi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, \Gamma_r^{t,x}) dr - \int_s^T Z_r^{t,x} dW_r \\ \qquad\qquad - \int_s^T \int_E U_r^{t,x}(e) \tilde{\mu}(de, dr). \end{cases}$$

Then, $u(t,x) = Y_t^{t,x}$, $(t,x) \in [0,T] \times \mathbb{R}^d$ is the unique viscosity solution of the PIDE [PP91, BBP97], and the triple $(Y_s^{t,x}, Z_s^{t,x}, U_s^{t,x})$ for $s \le t \le T$ can be represented by

$$\begin{cases} Y_s^{t,x} = u(s, X_s^{t,x}), \\ Z_s^{t,x} = \sigma(s, X_s^{t,x}) \nabla u(s, X_s^{t,x}), \\ U_s^{t,x} = u(s, X_{s-}^{t,x} + c(s-, X_{s-}^{t,x}, e)) - u(s, X_{s-}^{t,x}), \end{cases}$$

and $\Gamma_s^{t,x}$ is defined by $\Gamma_s^{t,x} = \mathcal{B}[u](s, X_s^{t,x})$.

Why we are interested in BSDEs?

Applications of BSDEs and KBE

- Differences between forward and backward Kolmogorov equations
  - Kolmogorov forward equation (KFE), i.e. Fokker-Plank equation: given a distribution about the state X_t at time t, i.e. $p(t, x)$, we want to know the probability distribution of the state at a later time $s > t$.
  - Kolmogorov backward equation (KBE): given a target function $u(T, x)$, we want to know, under the condition that X_t reaches (x, t) with $t \leq T$, what is the expectation of $u(T, X_T)$, i.e. $\mathbb{E}_t^x[u(T, X_T)|X_t = x]$.
  - The KBE of divergence form corresponds to another type of BSDEs where the forward process X_t is a Dirichlet type Markov process.
  - For BSDEs driven by pure jump processes, the corresponding KFE and KBE are of the same type.
- Applications of BSDEs
  - stochastic optimal control
  - mathematical finance, e.g. option pricing
  - solution of PDEs/PIDEs, e.g. hyperbolic conservation law, nonlocal diffusion

Numerical schemes for BSDEs driven by Brownian motion

- Backward stochastic differential equation:

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t dW_s, \\ Y_t = \varphi(X_T) + \displaystyle\int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \end{cases}$$

  where the solution is the pair (Y_t, Z_t).

- We do not need to discretize the forward SDE.
- We need to generate two reference equations for solving Y_t and Z_t.
- To overcome the low accuracy in discretizing the Itô integral, we propose to
  - remove the Itô integral in the reference equations
  - convert the Itô integral into a time integral in the reference equations

Pre-processing of the BSDE

- The reference equation for Y_t: taking the conditional mathematical expectation $\mathbb{E}_{t_n}^x[\cdot]$ on both sides of the BSDE, we obtain

$$Y_{t_n} = \mathbb{E}_{t_n}^x[Y_{t_{n+1}}] + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[f(s, X_s, Y_s, Z_s)]\, ds,$$

  where the Itô integral is removed due to the fact that $\int_t^T Z_s dW_s$ is a martingale such that $\mathbb{E}_{t_n}^x[\int_{t_n}^{t_{n+1}} Z_s dW_s] = 0$.

- The reference equation for Z_t: Multiply $\Delta W_{t_{n+1}}$ on both sides, take $\mathbb{E}_{t_n}^x[\cdot]$ and apply the Itô isometry formula, we obtain

$$0 = \mathbb{E}_{t_n}^x[Y_{t_{n+1}}\Delta W_{t_{n+1}}] + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[f(s, X_s, Y_s, Z_s)\Delta W_s]\, ds - \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[Z_s]\, ds,$$

  where the Itô integral is converted by

$$\mathbb{E}_{t_n}^x\left[\int_{t_n}^{t_{n+1}} Z_s dW_s \Delta W_{t_{n+1}}\right] = \mathbb{E}_{t_n}^x\left[\int_{t_n}^{t_{n+1}} Z_s dW_s \int_{t_n}^{t_{n+1}} 1 dW_s\right] = \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[Z_s]\, ds$$

Semi-discrete schemes

time discretization

- θ-scheme [ZCP06, ZLZ12]

$$
\begin{cases}
Y^n = \mathbb{E}_{t_n}^x\left[Y^{n+1}\right] + (1-\theta)\Delta t\, \mathbb{E}_{t_n}^x\left[f(t_{n+1}, X^{n+1}, Y^{n+1}, Z^{n+1})\right] \\
\qquad\qquad + \theta \Delta t f\left(t_n, X^n, Y^n, Z^n\right), \\
\theta\Delta t\, Z^n = \mathbb{E}_{t_n}^x\left[Y^{n+1}\Delta W_{t_{n+1}}\right] + (1-\theta)\Delta t\, \mathbb{E}_{t_n}^x\left[Z^{n+1}\right] \\
\qquad\qquad + (1-\theta)\Delta t\, \mathbb{E}_{t_n}^x\left[f(t_{n+1}, X^{n+1}, Y^{n+1}, Z^{n+1})\Delta W_{t_{n+1}}\right],
\end{cases}
$$

where $\theta = 0$: forward Euler scheme, $\theta = 1$: backward Euler scheme and $\theta = \frac{1}{2}$: Crank-Nicolson scheme.

- Other time-stepping schemes
  - linear multi-step scheme [ZZJ10]
  - generalized θ-scheme [ZLZ12]
  - second-order scheme for coupled BSDEs [ZZJ14]

A fully-discrete scheme

Estimation of high-dimensional conditional expectations

- At each point (t_n, x), we need to estimate

$$\mathbb{E}_{t_n}^{x}[Y^{n+1}] = \frac{1}{(2\pi\Delta t)^{d/2}} \int_{\mathbb{R}^d} Y^{n+1}(v) \exp\left[-\frac{(v-x)^\top (v-x)}{2\Delta t}\right] dv,$$

and $\mathbb{E}_{t_n}^{x}[f^{n+1}], \mathbb{E}_{t_n}^{x}[Y^{n+1}\Delta W_{t_{n+1}}], \mathbb{E}_{t_n}^{x}[f^{n+1}\Delta W_{t_{n+1}}], \mathbb{E}_{t_n}^{x}[Z^{n+1}]$.

- Sparse-grid quadrature rule based on Gauss-Hermite points can be directly used to discretize the conditional expectations, i.e.,

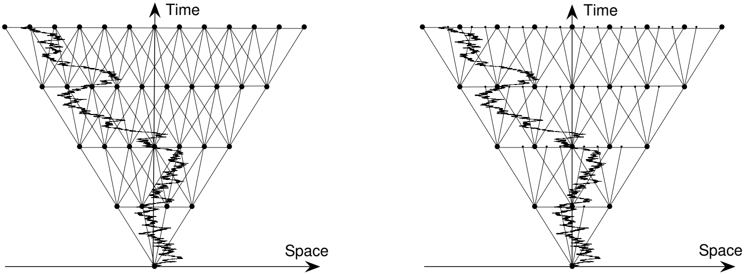
$$\mathbb{E}_{t_n}^{x}[Y^{n+1}] \approx \widehat{\mathbb{E}}_{t_n}^{x}[Y^{n+1}] = \sum_{i=1}^{Q} \omega_i Y^{n+1}(x + \sqrt{2\pi\Delta t} a_i),$$

where $\{w_i\}_{i=1}^{Q}$, $\{a_i\}_{i=1}^{Q}$ are weights and the sparse grid abscissa, respectively.

A fully-discrete scheme

high-dimensional Interpolation

- Instead of discretizing differential operators, at each time-space point (t_n, x), we need to compute a set of conditional expectations.



(left) The ideal case: the quadrature points are on the mesh ; (right) Gauss-Hermite case: most quadrature points are not on the mesh

- To avoid exponential growth of the total number of quadrature points, we construct an interpolant of the solutions Y^{n+1}, Z^{n+1} at each time level on a pre-determine mesh, and interpolate at all quadrature points.

A fully-discrete scheme

At each sparse-grid point x_i at the time t_n, the solution (Y_i^n, Z_i^n) is obtained by

$$\begin{cases} Y_i^n = \widehat{\mathbb{E}}_{t_n}^{x_i}\left[\widehat{Y}^{n+1}\right] + (1-\theta)\Delta t\, \widehat{\mathbb{E}}_{t_n}^{x_i}\left[f(t_{n+1}, X^{n+1}, \widehat{Y}^{n+1}, \widehat{Z}^{n+1})\right] \\ \qquad\qquad + \theta \Delta t f\Big(t_n, X_i^n, Y_i^n, Z_i^n\Big), \\ \theta \Delta t\, Z_i^n = \widehat{\mathbb{E}}_{t_n}^{x_i}\left[\widehat{Y}^{n+1}\Delta W_{t_{n+1}}\right] + (1-\theta)\Delta t\, \widehat{\mathbb{E}}_{t_n}^{x_i}\left[\widehat{Z}^{n+1}\right] \\ \qquad\qquad + (1-\theta)\Delta t\, \widehat{\mathbb{E}}_{t_n}^{x_i}\left[f(t_{n+1}, X^{n+1}, \widehat{Y}^{n+1}, \widehat{Z}^{n+1})\Delta W_{t_{n+1}}\right], \end{cases}$$

where $\widehat{\mathbb{E}}_{t_n}^{x_i}[\cdot]$ denotes the sparse Gauss-Hermite quadrature rule and $\widehat{Y}^{n+1}, \widehat{Z}^{n+1}$ are sparse polynomial interpolants.

- The multi-variate Gaussian distribution is defined on the unbounded domain $\mathbb{R}^d$ but the quadrature rule is always on a bounded domain, such that the sparse interpolation is only needed on the bounded domain.
- Possible choice of sparse interpolation: Clanshaw-Curtis rule [,NTW08], Leja [ddd], local hierarchal basis [BG04], wavelet basis [BG04,GWZ14]

The advantages of solving BSDEs instead of PDEs

- No linear system is involved when using implicit time-stepping scheme. The scheme for solving Z_t is always explicit.
  - If $f(t, X_t, Y_t, Z_t)$ is linear with respect to Y_t, i.e. linear PDEs, then the scheme for Y_t is also explicit.
  - If $f(t, X_t, Y_t, Z_t)$ is nonlinear with respect to Y_t, then a nonlinear equation has to be solved at each sparse grid point.
- The PDE can be solved independently at different sparse grid points on each time level, which makes it straightforward to incorporate parallelization and adaptivity.
- The stability of our scheme follows classic stability properties of the time-stepping schemes for deterministic ODEs.
- Challenge: it is generally difficult to construct high-order schemes for initial-boundary value problems because of the involved stopping time.

Numerical example 1

A multi-dimensional BSDE

We consider a d-dimensional BSDE with d from 2 to 4. Let $W_t = \left(W_t^1, \cdots, W_t^d\right)^\top$ be a d-dimensional Brownian motion. W_t^i $(i = 1, \cdots, d)$ are d independent standard one-dimensional Brownian motions. The BSDE of interest is

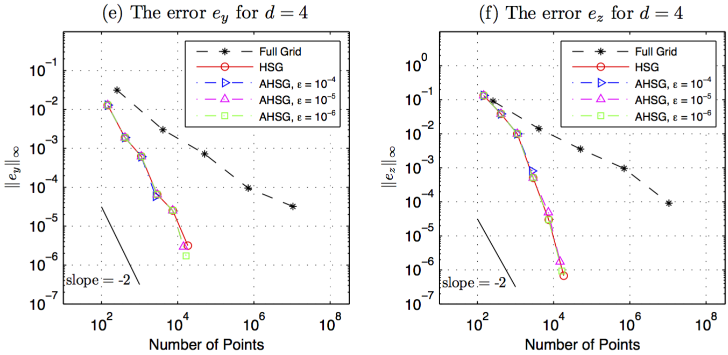
$$\begin{cases} -dY_t = \left[(d-1)Y_t + 2\sum_{i=1}^{d} W_t^i Z_t^i\right] dt - Z_t dW_t, \\ Y_T = \exp\left[T - \sum_{i=1}^{d}\left(W_T^i\right)^2\right], \end{cases}$$

where $Z_t = (Z_t^1, \cdots, Z_t^d)$. The analytical solution of (14) is given by

$$\begin{cases} Y_t = \exp\left[t - \sum_{i=1}^{d}(W_t^i)^2\right], \\ Z_t^i = -2W_t^i \exp\left[t - \sum_{i=1}^{d}(W_t^i)^2\right], \quad i = 1, \cdots, d. \end{cases}$$

Numerical example 1

A multi-dimensional BSDE



The errors of the approximations to Y_t (left) and Z_t (right) with respect to the number of sparse grid points. Linear hierarchal polynomial basis [BG04] is used.

Numerical example 2

European call option in the Black-Scholes model

Denote p_t and $S_t = (S_t^1, \dots, S_t^d)$ as the bond price and the prices of d independent stocks, respectively. Assume that p_t and S_t satisfy

$$\begin{cases} dp_t = r_t p_t dt, \\ dS_t^i = b_t^i S_t^i dt + \sigma_t^i S_t^i dW_t^i, \quad i = 1, \dots, d. \end{cases}$$

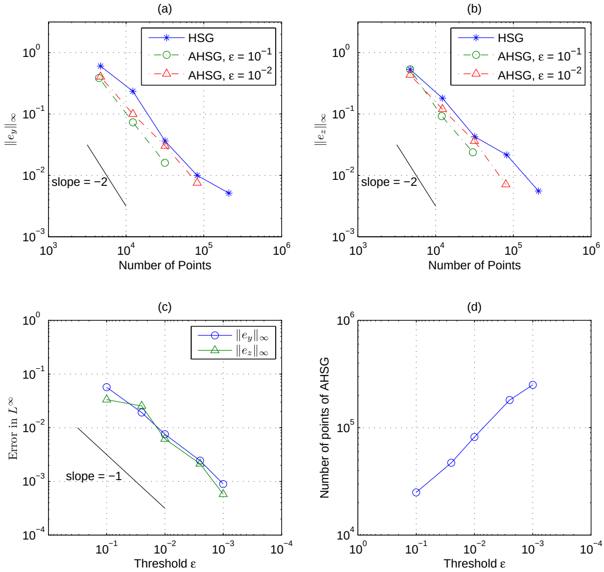
An investor with total wealth y_t at time t puts π_t^i money to buy the i-th stock and uses $y_t - \sum_{i=1}^d \pi_t^i$ to buy the bond. Then the processes y_t and π_t^i $(i = 1, \dots, d)$ satisfy the following BSDE [KPQ97]:

$$-dy_t = -\left[r_t y_t + \sum_{i=1}^d \frac{b_t^i - r_t + q_t^i}{\sigma_t^i} z_t^i \right] dt - \sum_{i=1}^d z_t^i dW_t^i,$$

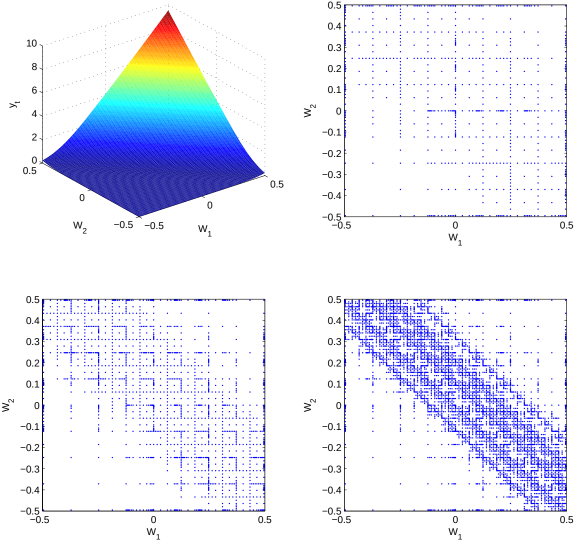
where $z_t = (z_t^1, \dots, z_t^d) = (\sigma_t^1 \pi_t^1, \dots, \sigma_t^d \pi_t^d)$ and the terminal condition for European call option is

$$y_T = \max \left\{ \prod_{i=1}^d (S_t^i)^{\alpha_i} - K, 0 \right\},$$

where $\alpha_i > 0$, $\sum_{i=1}^d \alpha_i = 1$, S_T is the solution of S_t at the mature time T and K is the strike price.



(a,b) The errors e_y and e_z with respect to the number of grid points for $d = 8$, $T = 10$; (c)the error e_y, e_z with respect to the threshold ε for the AHSG method; (d) the growth of the number of grid points with respect to the threshold in Example 2.



The evolution of the adaptive sparse grid at the time level t_{N-1} with the threshold $\varepsilon = 10^{-3}$ in Example 2.

BSDEs driven by Lèvy jump processes

motivated by nonlocal diffusion problems

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Consider the following BSDE driven by a compound Poisson process, i.e.,

$$\begin{cases} X_t = X_0 + \int_0^t \int_D e\, \mu(de, ds), \\ Y_t = \varphi(X_T) + \int_t^T f(s, X_s, Y_s) ds - \int_t^T \int_D U_s(e)\, \tilde{\mu}(de, ds). \end{cases}$$

It corresponds to the following nonlocal diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \mathcal{L}u(t,x) = g(t,x,u) \quad \text{for } (t,x) \in (0,T] \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \text{ for } x \in \mathbb{R}^d, \end{cases}$$

where the diffusion operator $\mathcal{L}$ is defined by

$$\mathcal{L}u = \int_D \big(u(t, x+e) - u(t,x)\big)\, \gamma(e)\, de, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^d,$$

and $g(T-t, x, u) = f(t,x,u)$, $\varphi(x) = u_0(x)$.

A semi-discrete scheme for Y_t

time discretization

Based on a time partition $\mathcal{T} = \{0 = t_0 < \cdots < t_N = T\}$, since the process $\int_{t_n}^{t} \int_D U_s(e)\tilde{\mu}(de, ds)$ for $t > t_n$ is a martingale, we have

$$Y_{t_n} = \mathbb{E}_{t_n}^x[Y_{t_{n+1}}] + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[f(s, X_s, Y_s)]ds.$$

The semi-discrete scheme is as follows: given random variable $Y_N = Y_{t_N} = \varphi(X_T)$ being the terminal condition, for $n = N-1, \ldots, 1, 0$, the solution Y_{t_n} at $X_n = x$ is approximated by Y_n satisfying

$$\begin{cases} X_{n+1} = X_n + \displaystyle\int_D e\,\mu(de, \Delta t), \\ Y_n = \mathbb{E}_{t_n}^x\left[Y_{n+1}\right] + (1-\theta)\Delta t_n \mathbb{E}_{t_n}^x\left[f(t_{n+1}, X_{n+1}, Y_{n+1})\right] \\ \qquad\qquad + \theta \Delta t_n f(t_n, X_n, Y_n), \end{cases}$$

where $0 \le \theta \le 1$.

- It looks very similar to the case with Brownian motions. However, the conditional expectation $\mathbb{E}_{t_n}^x[\cdot]$ is different because X_t is a jump process.

Compound Poisson processes

- A standard Poisson process, denoted by N_t, is a stochastic process with jumps of size $+1$ only. For fixed $0 \leq s \leq t$, we have

$$\mathbb{P}(N_t - N_s = k) = e^{-\lambda(t-s)} \frac{(\lambda(t-s))^k}{k!}$$

- A compound Poisson process, denoted by X_t, is a generalization of standard Poisson process by introducing random jump size, i.e.,

$$X_t = \sum_{k=1}^{N_t} Z_k,$$

where $\{Z_k\}_{k\geq 1}$ are i.i.d. sequence of square-integrable random variables with probability distribution $\rho(Z)$.

- The compensated Poisson random measure $\tilde{\mu}(de, dt)$ of X_t can be written as

$$\tilde{\mu}(de, dt) = \mu(de, dt) - \lambda(de) = \mu(de, dt) - \lambda\rho(e)de,$$

where $\lambda = \int_E \lambda(de) < \infty$ is the jump intensity.

Estimation of $\mathbb{E}_{t_n}^{x}[\cdot]$

When the BSDE is driven by a compound Poisson process X_t, then we have

$$\begin{aligned}
\mathbb{E}_{t_n}^{x_i}\left[Y_{n+1}\right] &= \sum_{m=0}^{\infty} \mathbb{P}\Big\{N_{t_{n+1}} - N_{t_n} = m\Big\}\, \mathbb{E}\Big[Y_{n+1}\Big(x_{\mathbf{i}} + \sum_{k=1}^{m} e_k\Big)\Big] \\
&= \sum_{m=0}^{\infty} \exp(-\lambda \Delta t_n)\, \frac{(\lambda \Delta t_n)^m}{m!}\, \mathbb{E}\Big[Y_{n+1}\Big(x_i + \sum_{k=1}^{m} e_k\Big)\Big] \\
&= \exp(-\lambda \Delta t_n)\, Y_{n+1}(x_i) + \sum_{m=1}^{\infty} \exp(-\lambda \Delta t_n)\, \frac{(\lambda \Delta t_n)^m}{m!} \\
&\quad \times \int_D \cdots \int_D Y_{n+1}\Big(x_i + \sum_{k=1}^{m} e_k\Big)\Big(\prod_{k=1}^{m} \rho(e_k)\Big) de_1 \cdots de_m.
\end{aligned}$$

When retaining the first M_y jumps, we have the following estimation

$$\begin{aligned}
\widehat{\mathbb{E}}_{t_n, M_y}^{x_i}\left[Y_{n+1}\right] &= \exp(-\lambda \Delta t)\, Y_{n+1}(x_i) \\
&\quad + \sum_{m=1}^{M_y} \exp(-\lambda \Delta t) \frac{(\lambda \Delta t)^m}{m!} \sum_{q=1}^{Q_m} w_q^m\, Y_{n+1}(x_i + |\boldsymbol{a}_q^m|),
\end{aligned}$$

A fully-discrete scheme

Error estimate

- In one-dimensional case, the fully-discrete scheme is given by

$$\begin{cases} X_{n+1} = x_i + \int_D e\,\mu(de, \Delta t), \\ Y^i_{n,p} = \widehat{\mathbb{E}}^{x_i}_{t_n, M_y}[Y_{n+1,p}] + (1-\theta)\Delta t_n \widehat{\mathbb{E}}^{x_i}_{t_n, M_f}[f(t_{n+1}, X_{n+1}, Y_{n+1,p})] \\ \qquad\qquad + \theta \Delta t_n f\left(t_n, X_n, Y^{\mathbf{i}}_{n,p}\right), \end{cases}$$

where $Y_{n+1,p}$ is constructed by p-th order Lagrange interpolation.

Theorem [ZZWG14]

Let Y_{t_n} and $Y^i_{n,p}$ for $n = 0, 1, \ldots, N$, $i \in \mathbb{Z}$ be the exact solution of the BSDE and the fully-discrete solution with $\theta = \frac{1}{2}$, respectively. Then, the error $e^i_n = Y^{x_i}_{t_n} - Y^i_{n,p}$ can be bounded by

$$\max_{i\in\mathbb{Z}} |e^i_n| \leq C\left[(\Delta t)^2 + (\lambda \Delta t)^{M_y} + (\lambda\Delta t)^{M_f+1} + Q^{-r} + (\Delta x)^{p+1}\right],$$

where the constant C only depends on the terminal time T, the jump intensity λ, the upper bounds of f and φ and their derivatives.

Numerical example 3

symmetric jump kernel

We consider the following nonlocal diffusion problem in $[0, T]$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{1}{\delta^3} \displaystyle\int_{-\delta}^{\delta} \Big(u(t, x+e) - u(t,x) \Big) de = g(t,x), \quad t > 0, \\ u(0,x) = \varphi(x), \end{cases}$$

where $\delta > 0$ and the symmetric kernel $\gamma(e)$ is defined as

$$\gamma(e) = \begin{cases} \dfrac{1}{\delta^3}, & \text{for } e \in [-\delta, \delta], \\ 0, & \text{for } e \notin [-\delta, \delta]. \end{cases}$$

We choose the exact solution to be

$$u(t,x) = (-x^3 + x^2) \exp\left(- \frac{t}{10} \right),$$

therefore the forcing term g is given by

$$g(t,x) = -\frac{u(t,x)}{10} + \left(2x - \frac{2}{3} \right) \exp\left(- \frac{t}{10} \right).$$

Time discretization error

Errors and convergence rates with respect to Δt where $T = 1$, $\delta = 1$, $N_x = 65$, $p = 3$

$\|Y_{t_0} - Y_{0,p}\|_\infty$						
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$	CR
$\theta = 0,\ M_y = 0,\ M_f = 0$	1.932E-1	1.978E-1	2.000E-1	2.012E-1	2.017E-1	-0.015
$\theta = 0,\ M_y = 1,\ M_f = 0$	7.958E-2	3.435E-2	1.572E-2	7.487E-3	3.649E-3	1.109
$\theta = 0,\ M_y = 2,\ M_f = 1$	3.673E-2	1.849E-2	9.279E-3	4.675E-3	2.326E-3	0.995
$\theta = 1,\ M_y = 0,\ M_f = 0$	2.111E-1	2.067E-1	2.045E-1	2.034E-1	2.028E-1	0.014
$\theta = 1,\ M_y = 1,\ M_f = 0$	4.891E-2	2.581E-2	1.328E-2	6.736E-3	3.393E-3	0.964
$\theta = 1,\ M_y = 2,\ M_f = 1$	6.170E-2	3.007E-2	1.468E-2	7.231E-3	3.585E-3	1.027
$\theta = \frac{1}{2},\ M_y = 0,\ M_f = 0$	2.030E-1	2.025E-1	2.023E-1	2.023E-1	2.023E-1	0.001
$\theta = \frac{1}{2},\ M_y = 1,\ M_f = 0$	2.623E-2	1.326E-2	6.666E-3	3.342E-3	1.674E-3	0.993
$\theta = \frac{1}{2},\ M_y = 2,\ M_f = 1$	3.207E-3	8.258E-4	2.094E-4	5.271E-5	1.322E-5	1.981
$\theta = \frac{1}{2},\ M_y = 3,\ M_f = 2$	3.330E-3	8.632E-4	2.196E-4	5.538E-5	1.391E-5	1.977

Numerical example 4

non-symmetric jump kernel and discontinuous solution

We consider the following nonlocal diffusion problem in $[0, T]$,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\int_{-\delta}^{2\delta} \Big[u(t, x+e) - u(t,x) \Big] de = g(t,x), & t > 0, \\ u(0,x) = \varphi(x), \end{cases}$$

where $\delta > 0$ and an non-symmetric kernel $\gamma(e)$ is defined as

$$\gamma(e) = \begin{cases} 1, & \text{if } e \in [-\delta, 2\delta], \\ 0, & \text{if } e \notin [-\delta, 2\delta]. \end{cases}$$

We choose the exact solution to be

$$u(t,x) = \begin{cases} x \sin(t), & \text{if } \ x < \dfrac{1}{2}, \\ x^2 \sin(t), & \text{if } \ x \geq \dfrac{1}{2}, \end{cases}$$

and g can be computed accordingly.

Numerical example 4

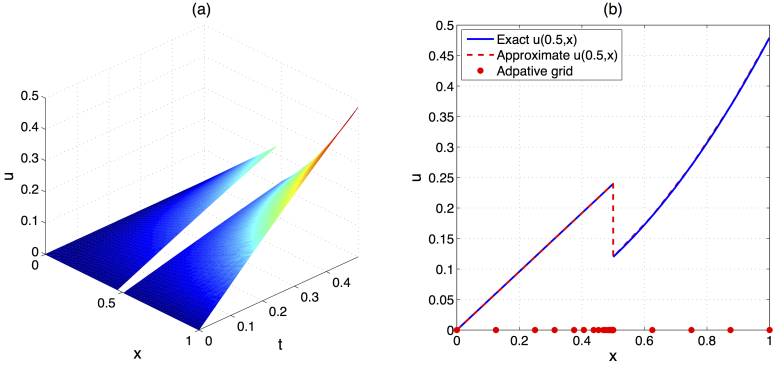
non-symmetric jump kernel and discontinuous solution

Errors and convergence rates with respect to Δx where $p = 1$, $T = 0.5$, $\theta = \frac{1}{2}$, $N = 512$.

	$\delta = 1$		$\delta = 0.1$	
Δx	$\|Y_{t_0} - Y_{0,p}\|_{L^2}$	$\|Y_{t_0} - Y_{0,p}\|_{L^\infty}$	$\|Y_{t_0} - Y_{0,p}\|_{L^2}$	$\|Y_{t_0} - Y_{0,p}\|_{L^\infty}$
2^{-3}	3.001E-02	1.323E-01	2.503E-02	1.215E-01
2^{-4}	2.287E-02	1.317E-01	1.751E-02	1.207E-01
2^{-5}	1.467E-02	1.257E-01	1.231E-02	1.201E-01
2^{-6}	1.107E-02	1.254E-01	8.676E-03	1.197E-01
2^{-7}	7.045E-03	1.221E-01	6.126E-03	1.192E-01
CR	0.523	0.030	0.507	0.007

Numerical example 4

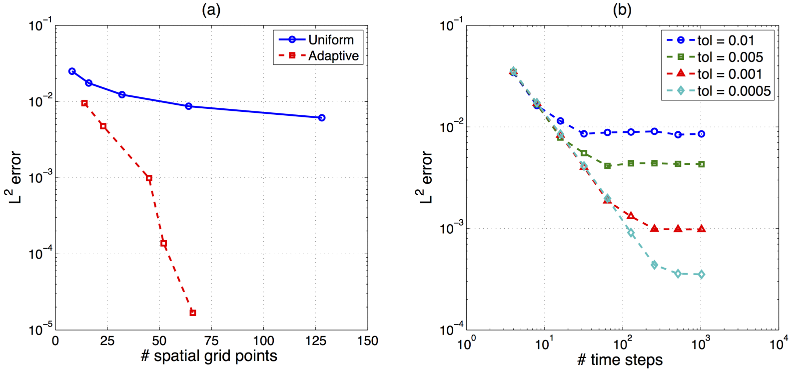
non-symmetric jump kernel and discontinuous solution



(a) The surface of $u(t,x)$ in $[0,0.5]\times[0,1]$; (right) The exact solution $u(0.5,x)$ (solid line) and its approximation (dashed line) using 33 grid points (red dots).

Numerical example 4

non-symmetric jump kernel and discontinuous solution



(a) Convergence with respect to number of spatial grid points using uniform and adaptive grids, respectively. (b) Convergence with respect to Δt for various tolerances of adaptive spatial grid.

Concluding remarks

- Backward SDE is advantageous for numerical solution of a class of partial-integral differential equations, especially when integral operator is involved.
- Nonlinear system solution is involved for implicit time-stepping schemes
- It is easy to incorporate adaptive approximation and parallel implementation
- The sparse-grid method is used for approximating conditional expectation and interpolation of dimension M, where M is
  - for pure Brownian motion, $M = d$
  - for a general continuous diffusion processes, $M = d$ for first-order schemes, $M = 2d$ for second-order schemes
  - for pure jump process, $M = d\times$ number of jumps
- For future works, we will extend our approach to
  - BSDEs driven by *infinite activity* Lèvy processes ($\lambda = \infty$) which cannot be represented by compound Poisson processes, so new sparse-grid quadrature rules are needed.
  - PIDEs with discontinuous solutions where the discontinuity location moves as time evolves.

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